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93202Q

932022

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

Scholarship 2007
Mathematics with Calculus

9.30 am Saturday 1 December 2007
Time allowed: Three hours
Total marks: 40

QUESTION BOOKLET

A 4-page booklet (S-CALCF) containing mathematical formulae and tables has been centre-stapled in the middle of this booklet. Before commencing, carefully detach the Formulae and Tables Booklet and check that none of its pages is blank.

Answer ALL questions.

Write ALL your answers in the Answer Booklet 93202A.

Show ALL working. Start each question on a new page. Number each question carefully.

Check that this Question Booklet 93202Q has pages 2–6 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

You have three hours to complete this examination.

QUESTION ONE (8 marks)

An icemaker produces ice in the shape of paraboloids that may be modelled by rotating the graph of $y^2 = 4ax$ through 360° about the x-axis (see Fig. 1).

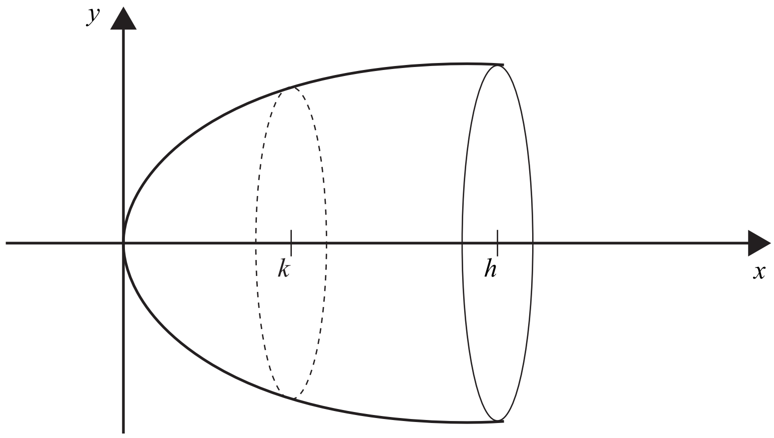

Fig. 1

(a) Find, in terms of a and h, the volume of an ice paraboloid of length h.

(b) The surface area of a solid of revolution such as that in Fig. 1, is given by:

$$\text{surface of revolution} = \int_0^h y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Find, in terms of h, the area of the surface of revolution of an ice paraboloid of length h.

(c) Each paraboloid piece of ice produced melts such that the rate of change of its volume is proportional to the area of 'surface of revolution' of the ice.

If the length h is changing at a rate of 0.26 mm/sec when $h = 8a$, find the rate at which the volume of a piece of ice is changing when $h = 3a$, in terms of a and π.

QUESTION TWO (8 marks)

(a) Solve for $0 \le x \le 2\pi$.

$$10\pi\cos\left(\frac{\pi x}{2}\right)+10\pi\cos\left(\frac{2\pi x}{3}\right)=0.$$

(b) An amusement park has a giant double Ferris wheel as shown in Fig. 2.

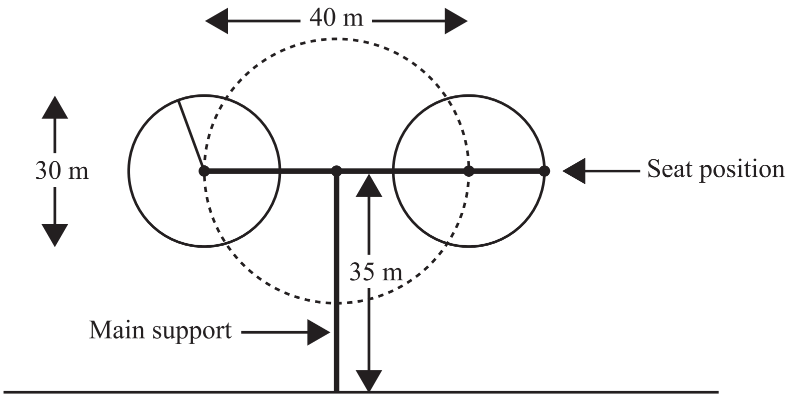

Fig. 2

The double Ferris wheel has a rotating arm 40 metres long attached at its centre to a main support 35 metres above the ground. At each end of the rotating arm is attached a Ferris wheel measuring 30 metres in diameter, as shown in the diagram. The rotating arm takes 4 minutes to complete one full revolution, and each wheel takes 3 minutes to complete a revolution about that wheel's hub. All revolutions are anticlockwise, in a vertical plane.

At time $t = 0$ the rotating arm is parallel to the ground and your seat is at the 3-o'clock position of the rightmost wheel.

(i) Find a formula for $h(t)$, your height above the ground in metres, as a function of time in minutes.

(ii) Hence find your maximum height above the ground, clearly showing that it is a maximum.

QUESTION THREE (8 marks)

The graphs of the two functions

$$f(x) = 3\sin 2x \quad 0 \le x \le 2\pi \text{ and } g(x) = 2\cos x \quad 0 \le x \le 2\pi$$

are shown in Fig. 3.

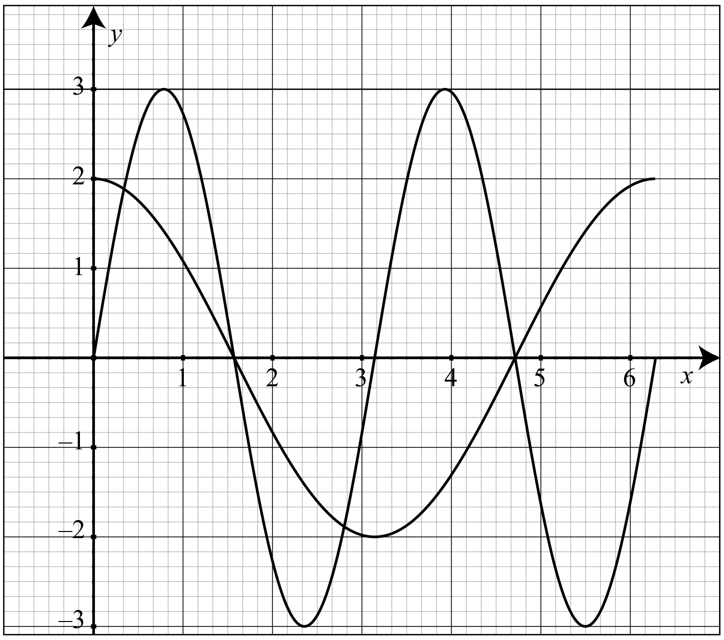

Fig. 3

(a) $h(x) = k + 2\cos x \quad 0 \le x \le 2\pi$, where k is a positive constant.
Find the x co-ordinates of all the points at which the graphs of $f(x)$ and $h(x)$ have the same gradient.

(b) The graph of $p(x) = -a(x-b)^2 - c \quad 0 \le x \le 2\pi$, where $a > 0$, $b > 0$, and $c > 0$, intersects each of the graphs of $f(x)$ and $g(x)$ at exactly one point only.
The x co-ordinate of the point of intersection with $f(x)$ is 2.575 and the vertex of $p(x)$ lies on $g(x)$.
Find the values of a, b and c.

(c) Find the range of values of k for which $f(x)$ and $h(x)$ intersect at exactly two points.
Note: $k > 0$ and both $f(x)$ and $h(x)$ have domain $0 \le x \le 2\pi$

QUESTION FOUR (8 marks)

(a) If $z = \sqrt{\frac{1}{2}\left(a + \sqrt{(a^2+b^2)}\right)} + \mathrm{i}\sqrt{\frac{1}{2}\left(-a + \sqrt{(a^2+b^2)}\right)}$ is a complex number, with $\mathrm{i} = \sqrt{-1}$ and a, b real numbers, find z^2 in the form $p + \mathrm{i}q$.

The Mandelbrot set (see Fig. 4) is constructed by plotting in black all complex numbers, $c = a + \mathrm{i}b$, such that

$$\text{if } f(z) = z^2 + c \quad \text{then} \quad \left|f^n(0)\right| < 2 \text{ for all } n \in \{1,2,3,...\}$$

where $\left|f^n(z)\right|$ represents the modulus of the complex number $\underbrace{f(f(f(f(...f(z)))))}_{n}$.

That is, $f^2(z) = f(f(z)) = (z^2 + c)^2 + c$. The sequence starts with $z = 0$, so $f(0) = c$, $f^2(0) = c^2 + c$, etc.

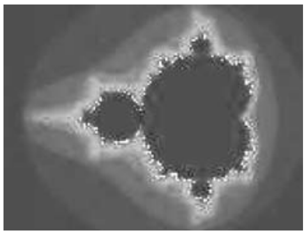

Fig. 4

(b) Use the definition of the Mandelbrot set above for $f(z) = z^2 + c$ beginning with $z = 0$, so $f(0) = c$.

(i) Show that $c = 1 + \frac{\mathrm{i}}{2}$ is not part of the Mandelbrot set, but $c = \mathrm{i}$ is.

(ii) For $f(z) = z^2 + c$, with $c = \sqrt{\frac{1}{2}\left(a + \sqrt{(a^2+b^2)}\right)} + \mathrm{i}\sqrt{\frac{1}{2}\left(-a + \sqrt{(a^2+b^2)}\right)}$ and $b^2 = 3a^2$, find $f^2(0)$ in terms of a.

Hence show that when $a = \frac{1}{8}$, $\left|f^2(0)\right| = \frac{1}{4}\sqrt{\left(5 + 2\sqrt{3}\right)}$.

QUESTION FIVE (8 marks)

A surfing company logo has the basic shape shown in Fig. 5 below. The two curves have parametric equations of the form:

Curve 1: $x = at^2 + b,\ y = 2at$ and Curve 2: $x = 3b - as^2,\ y = \sqrt{2}as,$

respectively, where a and b are constants.

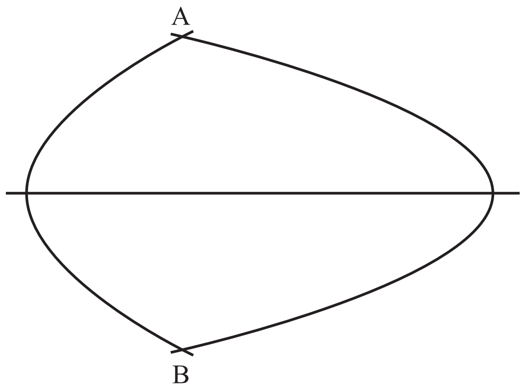

Fig. 5

(a) Find, in terms of a and b, the co-ordinates (x, y) of the two points of intersection, A and B, of the curves.

(b) The chord AB divides the area enclosed by the curves into two parts. Find the ratio of the two areas.

(c) The logo is enhanced by the addition of Curve 3 as shown in Fig. 6. This curve, which is the locus of the point M, is obtained by:

- taking the normal to Curve 2 at the point $\mathrm{P} = \left(3b - as^2, \sqrt{2}as\right)$;
- finding the point Q where the normal at P meets the x-axis;
- taking the mid-point M of the chord PQ;
- drawing the curve through all the points M for different values of s.

Show that the Cartesian equation of Curve 3 has the form $y^2 = ka(x + ma + nb)$, where k, m, and n are constants, and find the values of k, m, and n.

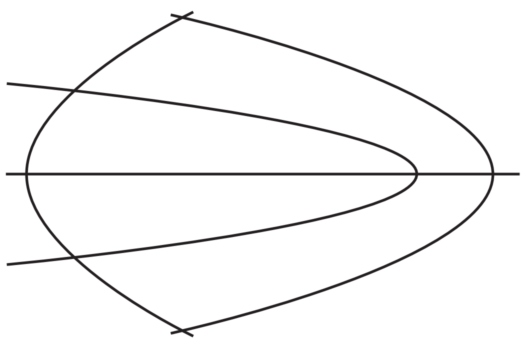

Fig. 6

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